Year 12 Lecture Series
Topic 10 - Proofs

Inspired Learning
Mr A S Gill

Topic 10 - Proofs

- Notation & Symbols
- Formal proof
- Disproof by counter example
- Proof by exhaustion

Prerequisites:
Topic 1 – Algebra

Topic 10 - Proofs

Notation* & *Symbols

$\mathbb{R}$: *The set of* ***all real numbers***.

$\mathbb{Z}$: *The set of* ***integers***

$\mathbb{Z}^+$: *The set of* ***positive integers*** (*where* 0 *is not considered positive*).

$\in$: *means* ***is a member of*** *e.g.* "$x \in \mathbb{Z}$" *means* ***x is a member of the set of integers***, *or* ***x is an integer***.

$\forall$: *means* ***for all***, *e.g.* "$\forall\ x \in \mathbb{Z}$" *means* ***for all real integer values of x***.

Topic 10 - Proofs

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Along with $\Rightarrow, \Leftarrow,$ *and* $\Leftrightarrow$ *to show implications.*

e.g. $x = 8 \Rightarrow x^2 = 64$

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$Decide\ whether\ the\ following\ statements\ are\ always, sometimes, or\ never\ true$:

$a)\ x^2 \geq x, x \in \mathbb{R}$

$b)\ x^3 \geq x, x \in \mathbb{Z}^+$

$c)\ n^2 + 1\ is\ an\ odd\ integer, n \in \mathbb{Z}$

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$Decide\ whether\ the\ following\ statements\ are\ always, sometimes, or\ never\ true:$

$x \in \mathbb{R} \Rightarrow 2x + 1$ is an odd integer

$n \in \mathbb{Z} \Rightarrow 2n + 1$ is an odd integer

$n^2 \geqslant 0 \Rightarrow n^3 \geqslant 0$

$n^2 + 1$ is an even integer $\Rightarrow n^3 + 1$ is an even integer

$x^2 \in \mathbb{Z} \Rightarrow x \in \mathbb{Z}$

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Given that n is a positive integer, determine whether each of the following statements is always true (A), sometimes true (S) or never true (N).

i $2n + 6$ is an even integer

ii $5n + 1$ is an odd integer

iii $n^2 + 1$ is an odd integer

Show that the following statement is false.

$$x - 7 = 0 \Leftrightarrow x^2 = 49$$

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Formal Proof

- $2n$ *is used to represent any even number.*
- $2n + 1$ *is used to represent any odd number.*
- $x^2 \geq 0, \ \forall\, x \in \mathbb{Z}$

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Prove that the sum of any two odd numbers is always even.

Prove that $a^2 + 6a + 10 > 0$ *for all real values of* a.

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Prove that the expression $(n+5)^2 - n^2$ *is odd for all integer values of* n.

Prove that there is no real value of λ such that the following equation has the real roots:

$$x^2 + (\lambda - 3)x + \lambda^2 + 5 = 0$$

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Proof by counter example

Prove by counter example that the inequality $x^2 + 4x + 3 > 0$ *is not true for every real value of* x.

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a Prove that for all real values of a and b

$$2ab \leqslant a^2 + b^2$$

b Sami believes that this inequality can be extended for all real values of a, b and c to give the following result:

$$3abc \leqslant a^2 + b^2 + c^2$$

Prove, by counter example, that she is wrong.

a Prove that $\forall x \in \mathbb{R}$

$$x^2 + 1 \geqslant 2x$$

b Prove by counter example that the inequality

$$x^2 + 1 > 2x$$

is not true for all real values of x.

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Proof by exhaustion

Prove by exhaustion that each of the first ten square numbers is either a multiple of 4*, or one more than a multiple of* 4.

Rohan believes there is always a prime number in between any two consecutive square numbers. Prove, by exhaustion, that this is true for the first ten square numbers (starting from 1^2).

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i Simplify $(n + 3)^2 - n^2$. Hence prove that, when n is an integer, $(n + 3)^2 - n^2$ is never an even number.

ii When n is an integer, is the expression $(n + 6)^2 - n^2$ always an even number, sometimes an even number, or never an even number? Explain your answer fully.

Show that $\forall n \in \mathbb{Z}$, the expression below is a multiple of 20

$$(2n + 5)^2 - (2n - 5)^2$$

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a Prove by exhaustion that every even number between 30 and 40 inclusive can be written as the sum of two primes.

b Prove by counter example that not **every** even number can be written as the sum of two primes.

c Beth believes that no odd number can be written as the sum of two primes. Prove by counter example that she is wrong.

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Summary:

- $2n \quad \textit{is used to represent any even number}.$
- $2n + 1 \textit{ is used to represent any odd number}.$
- $x^2 \geq 0, \quad \forall\, x \in \mathbb{Z}$
- $\textit{Proof by exhaustion involves working out all possible outcomes}.$
- $\textit{Proof by counter} - \textit{example only requires one contrary outcome}.$