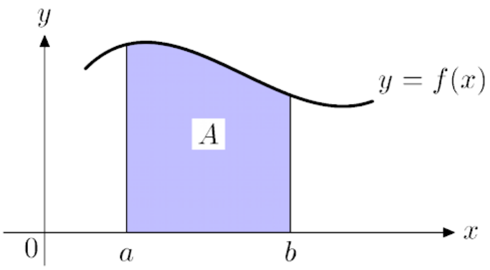
In this chapter we will explore a variety of new techniques for integration, as well as how integration can be applied. Such as:

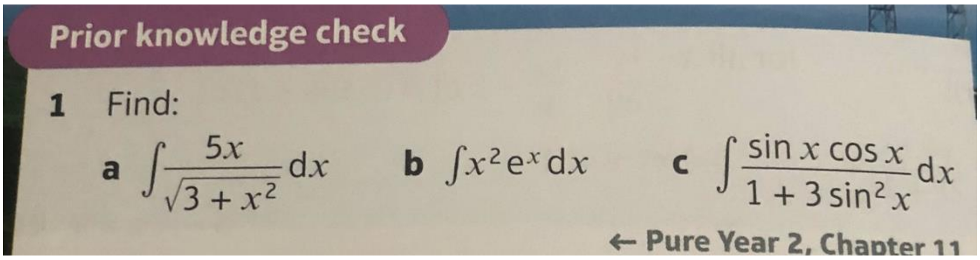
1) **Improper Integrals,**
2) **Mean value of a function,**
3) **Differentiating and integrating trigonometric functions**
4) **Integrating using partial fractions**

Prerequisite work- Definite Integrals

So far, we have a definition of integration when our limits of integration are finite, and the function is bounded



$$A = \int_a^b f(x)\,dx$$



Prior knowledge check

1 Find:

a $\int \frac{5x}{\sqrt{3+x^2}}\,dx$ **b** $\int x^2 e^x\,dx$ **c** $\int \frac{\sin x \cos x}{1+3\sin^2 x}\,dx$

← Pure Year 2, Chapter 11

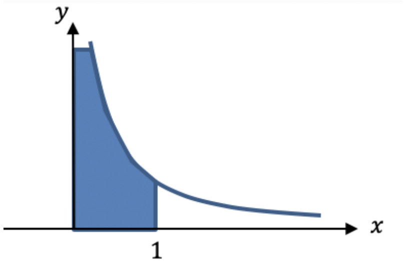
a)	b)	c)

Improper Integrals

Definition: Is an integral for which both or one of the limits is infinite, or the functions bounded.

Example 1 (unbounded)

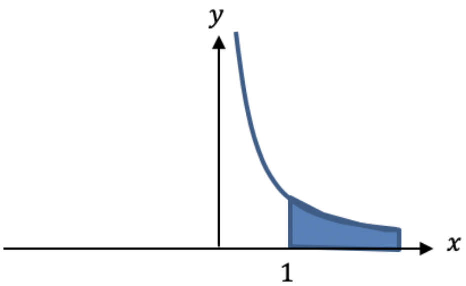
The integral $\int_0^1 \frac{1}{x}$ is an improper integral because 1/x is bounded on the interval [0,1]



Note: [0,1] means a closed interval therefore the integral includes all values from 0 to 1 including 0 and 1

Example 2 (infinite limits)

The integral $\int_1^\infty \frac{1}{x}$ is an improper integral because the upper limit of integration is infinite



Improper Integrals- Convergent and Divergent

The next step we need to consider is if your improper integral is convergent or divergent. What does this mean to us?

Convergent and Divergent using the infinite limit part of our improper integral definition

Definition: An improper integral of the form $\int_a^\infty f(x)$ dx converges if the integral $\int_a^r f(x)$ dx tends to a limit as r $\longrightarrow \infty$ (as r tends to infinity).

Example:

Evaluate the improper integral or show that it is not convergent (therefore divergent)

The improper integral $\int_1^\infty \frac{1}{x^2}$ dx

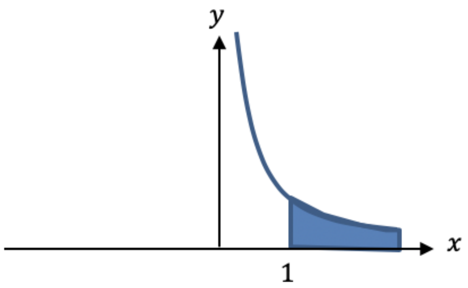
To see this, we evaluate $\lim_{r \to \infty} \int_1^r \frac{1}{x^2}$

$= \lim_{r \to \infty} [x^{-1}]_1^r$

$= \lim_{r \to \infty} (\frac{-1}{r} + 1)$

$= 1$

We know $\frac{1}{r} \to 0$ as $\frac{1}{r} \to \infty$ (denominator is getting very large as r goes to infinity so 1/r is getting very small).



Hence the improper integral converges to 1

Convergent and Divergent using the unbounded part of our improper integral definition

Definition: An improper integral in the form $\int_a^b f(x)\,dx$ with f(x) unbounded near a converges if $\int_r^b f(x)\,dx$ tends to a limit as r $\longrightarrow$ a where the variable r is taken to be in (a,b).

Example:

Evaluate the improper integral or show that it is not convergent (therefore divergent)

The improper integral $\int_0^1 \frac{1}{\sqrt{x}}\,dx$

To see this, we evaluate $\lim_{r \to 0} \int_r^1 \frac{1}{\sqrt{x}}\,dx$

$\lim_{r \to 0} \quad [2x^{\frac{1}{2}}]_r^1$

$\lim_{r \to 0} \quad (2-2\sqrt{r})$

we know that $\sqrt{r} \to 0$ as $r \to 0$, so $\int_r^1 \frac{1}{\sqrt{x}}\,dx \to 2$

Hence the improper integral converges to 2

Question

Using either the first definition for convergence and divergence or the second definition for convergence and divergence. Evaluate the improper integral showing if it is convergent or divergent.

The improper integral $\int_\infty^1 \frac{1}{x}\,dx$

Please complete the following exercises:
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When both integrals are infinite

If both limits of an integral are infinite, then you split the integral into the sum of two improper integrals

You need to check both integrals converge individually before you know the original integral converges.

$$\int_{\infty}^{-\infty} f(x)dx = \int_{\infty}^{c} f(x)dx + \int_{c}^{\infty} f(x)dx$$

Test your understanding

$$\int_{\infty}^{\infty} xe^{-x^2}dx$$

The mean value of a function

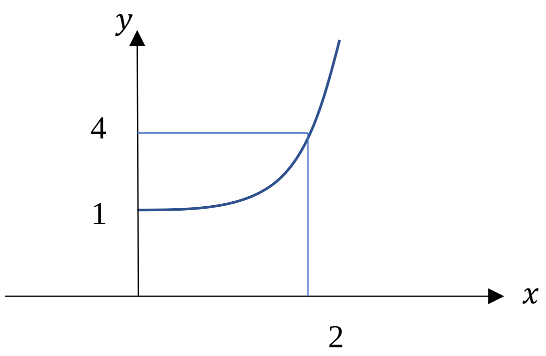
Currently we can find the mean set of values of a discrete set, however if we extend this to a continuous function f(x) between two limits x=a and x= b how could we calculate this?

The mean value of a function y=f(x) over the interval [a,b] is given by:

$$\frac{1}{b-a}\int_a^b f(x)dx$$

Example (to complete)

Say we had the function $f(x)=x^2+1$ and we wanted to find the average value of our function f(x) over the closed interval [0,3].



N.B our average function value will always correspond to an x value in our interval [a,b] as demonstrated from our sketch.

A few tricks when we have transformations of mean value functions

If the function f(x) has mean value **f** over the interval [a,b] and k a real constant then:

- f(x) + k has mean value **f** + k over the interval [a,b]
- kf(a) has mean value k**f** over the interval [a,b]
- -f(x) has mean value -**f** over the interval [a,b]

Example

$$f(x) = \frac{4}{1+e^{x}}$$

a) Show that mean value of f(x) over the interval [ln2, ln6] is $\frac{4ln\frac{9}{7}}{ln3}$

b) Use the answer to part a to find the mean value over the interval [ln2, ln6] of f(x)+4

Please complete the following questions:
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Differentiating with inverse trigonometric functions

By convention instead of writing $\sin^{-1}$ $\cos^{-1}$ $\tan^{-1}$ we write these functions as arcsin(x), arccos(x) and arctan(x).

Example differentiating arcsin(x) using implicit differentiation

let y= arcsinx

if we multiply by sin on both sides

siny= sin(arcsin(x)

siny=x (at this stage we use implicit differentiation)

cosy dy/dx = 1

dy/dx = 1/cosy (use trig identities)

dy/dx = $\frac{1}{\sqrt{1-sin^2 y}}$ (we know siny = x)

We know $\cos^2 y + \sin^2 y = 1$
$\cos^2 y = 1 - \sin^2 y$
cosy = $\sqrt{1 - sin^2 y}$

therefore

dy/dx = $\frac{1}{\sqrt{1-x^2}}$

We should learn the following

$$\frac{\mathbf{d}}{\mathbf{dx}}(\boldsymbol{arcinx}) = \frac{\mathbf{1}}{\sqrt{\mathbf{1}-\boldsymbol{x}^{\mathbf{2}}}}$$

$$\frac{\mathbf{d}}{\mathbf{dx}}(\boldsymbol{arccosx}) = -\frac{\mathbf{1}}{\sqrt{\mathbf{1}-\boldsymbol{x}^{\mathbf{2}}}}$$

$$\frac{\mathbf{d}}{\mathbf{dx}}(\boldsymbol{arctanx}) = \frac{\mathbf{1}}{\sqrt{\mathbf{1}+\boldsymbol{x}^{\mathbf{2}}}}$$

Example (to complete)

Given y= arcsinx^2 find dy/dx in terms of x

Please complete the following questions:
Ex 3C page 64 even questions

Integrating with inverse trigonometric functions

We can use trigonometric functions and identities to integrate integrals in form:

$\frac{1}{a^2+x^2}$ and $\frac{1}{\sqrt{a^2-x^2}}$

Example

Using substitution show that $\int \frac{1}{\sqrt{a^2-x^2}}$ dx = arcsin$(\frac{x}{a}\} + c$

Where mod x < a and a is positive.

Method 1

$\int \frac{1}{\sqrt{a^2-x^2}}$ dx = $\int \frac{1}{\sqrt{a^2(1-(\frac{x}{a}\}^2}}$ dx

= $\frac{1}{a}\int \frac{1}{\sqrt{(1-(\frac{x}{a}\}^2}}$ dx

Now let u=$\frac{x}{a}$
$\frac{du}{dx} = \frac{1}{a}$
du a = dx

= $\frac{1}{a}\int \frac{1}{\sqrt{(1-(\frac{x}{a}\}^2}}$ dx

= $\frac{1}{a}\int \frac{1}{\sqrt{(1-(u\}^2}}$ a du

= $\int \frac{1}{\sqrt{(1-(u\}^2}}$ du

From there we have to recognise this integral integrates to arcsin u +c

= arcsin u + c

= arcsin $(\frac{x}{a}) + c$

Method 2 (using trig identities)

$\int \frac{1}{\sqrt{a^2-x^2}}$ dx = $\int \frac{1}{acos\theta}$ × acosθdθ

=$\int 1\, d\theta$

= $\theta + c$ (rearrange for theta)

= arcsin$(\frac{x}{a}) + c$

Let x = a sin θ
$\frac{dx}{d\theta}$ = acosθ

dx = acosθdθ

and $\sqrt{a^2-x^2}$ = a$\sqrt{1-sin^2}\theta$

= a cosθ

Example

$\int \frac{1}{a^2+x^2}$ (think about what trig identity we would have to use here)

Test your understanding

$\int \frac{1}{25+x^2}\,dx$

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Integrating using partial fractions

$$\int \frac{1}{a^2-x^2}\,dx$$

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Summary of Integrals

INTEGRAL	INTEGRATED	HOW
$\frac{1}{\sqrt{a^2-x^2}}$	$\arcsin\left(\frac{x}{a}\right)+c$	trig identities
$\frac{1}{a^2+x^2}$	$\frac{1}{a}\arctan\left(\frac{x}{a}\right)+c$	trig identities
$\frac{1}{a^2-x^2}$	$\frac{1}{2}\ln\left(\frac{a+x}{a-x}\right)+c$	partial fractions
$\frac{1}{x^2-a^2}$	$\frac{1}{2}\ln\left(\frac{x-a}{x+a}\right)+c$	partial fractions